**Using completing square to simplify surds**

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**(1)** If $x > y > 0$, then : $\sqrt{(x+y) \pm 2\sqrt{xy}} = \sqrt{(\sqrt{x})^2 \pm 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2} = \sqrt{(\sqrt{x} \pm \sqrt{y})^2}$

$$= \sqrt{x} \pm \sqrt{y} \qquad \ldots. \quad \textbf{(1)}$$

**Example :** $\sqrt{5-\sqrt{24}} = \sqrt{3+2-2\sqrt{6}} = \sqrt{(\sqrt{3})^2 - 2\sqrt{3}\sqrt{2} + (\sqrt{2})^2} = \sqrt{(\sqrt{3}-\sqrt{2})^2} = \underline{\underline{\sqrt{3}-\sqrt{2}}}$

**(2)** If we put $x + y = A$, $xy = B$ in **(1)** , then

$A^2 = (x+y)^2 = x^2 + 2xy + y^2$

$\therefore \quad A^2 - 4B = (x^2 + 2xy + y^2) - 4xy = x^2 - 2xy + y^2 = (x-y)^2$

Since $x > y > 0$, $\sqrt{A^2 - 4B} = x - y$

$\therefore \quad x = \dfrac{(x+y)+(x-y)}{2} = \dfrac{A + \sqrt{A^2-4B}}{2}$ and $y = \dfrac{(x+y)-(x-y)}{2} = \dfrac{A - \sqrt{A^2-4B}}{2}$

Put all these in **(1)** , that is,

$$\sqrt{(x+y) \pm 2\sqrt{xy}} = \sqrt{x} \pm \sqrt{y}$$

we have the result:

If $A > 0$, $B > 0$ and $A^2 - 4B$ is a **perfect square**, then :

$$\sqrt{A \pm 2\sqrt{B}} = \sqrt{\frac{A+\sqrt{A^2-4B}}{2}} \pm \sqrt{\frac{A-\sqrt{A^2-4B}}{2}} \qquad \ldots. \quad \textbf{(2)}$$

It is undesirable to memorize this formula, but rather understand the use of completing square, as in the following example:

$$\sqrt{6-\sqrt{27}} = \sqrt{\frac{12-2\sqrt{27}}{2}} = \sqrt{\frac{(9+3)-2\sqrt{9\times 3}}{2}} = \sqrt{\frac{(\sqrt{9})^2 - 2\sqrt{9}\sqrt{3} + (\sqrt{3})^2}{2}}$$

$$= \sqrt{\frac{(\sqrt{9}-\sqrt{3})^2}{2}} = \frac{\sqrt{9}-\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}(\sqrt{9}-\sqrt{3})}{\sqrt{2}\sqrt{2}} = \underline{\underline{\frac{3\sqrt{2}-\sqrt{6}}{2}}}$$

**(3)** If $A^2 - 4B$ is **not** a perfect square, we can still use **(2)** , but we cannot simplify expression.

**Example** $\sqrt{8+2\sqrt{6}} = \sqrt{\dfrac{8+\sqrt{8^2-4(6)}}{2}} - \sqrt{\dfrac{8-\sqrt{8^2-4(6)}}{2}}$ , by **(2)**

$$= \sqrt{\frac{8+\sqrt{40}}{2}} - \sqrt{\frac{8-\sqrt{40}}{2}} = \sqrt{4+\sqrt{10}} + \sqrt{4-\sqrt{10}}$$

**(4) More examples :**

**(a)** Simplify $y=\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$

$$y = \sqrt{2^2+2\sqrt{4\times 3}+\sqrt{3}^2}+\sqrt{2^2-2\sqrt{4\times 3}+\sqrt{3}^2}$$

$$= \sqrt{\left(2+\sqrt{3}\right)^2}+\sqrt{\left(2-\sqrt{3}\right)^2}$$

$$= \left(2+\sqrt{3}\right)+\left(2-\sqrt{3}\right)=\underline{\underline{4}}$$

**(b)** Simplify $y=\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$

**(i)** If $x \ge 2$,

$$y = \sqrt{\left(\sqrt{x-1}\right)^2+2\sqrt{(x-1)\times 1}+1^2}+\sqrt{\left(\sqrt{x-1}\right)^2-2\sqrt{(x-1)\times 1}+1^2}$$

$$= \sqrt{\left(\sqrt{x-1}+1\right)^2}+\sqrt{\left(\sqrt{x-1}-1\right)^2}$$

$$= \left(\sqrt{x-1}+1\right)+\left(\sqrt{x-1}-1\right)=2\sqrt{x-1}$$

**(ii)** If $1 \le x < 2$ ,

$$y = \sqrt{1^2+2\sqrt{(x-1)\times 1}+\left(\sqrt{x-1}\right)^2}+\sqrt{1^2-2\sqrt{(x-1)\times 1}+\left(\sqrt{x-1}\right)^2}$$

$$= \sqrt{\left(1+\sqrt{x-1}\right)^2}+\sqrt{\left(1-\sqrt{x-1}\right)^2}$$

$$= \left(1+\sqrt{x-1}\right)+\left(1-\sqrt{x-1}\right)=2$$

$$\therefore \quad y=\begin{cases} 2\sqrt{x-1} & ,\text{if } x \ge 2 \\ 2 & ,\text{if } 1 \le x < 2 \end{cases}$$

Note : If $x < 1$ , $y$ is undefined.

**(5) Exercise :**

Simplify :

**(a)** $\dfrac{18+8\sqrt{3}}{2\sqrt{3}+\sqrt{12}-6\sqrt{3}}$

**(b)** $\sqrt{\dfrac{\frac{1}{2}\sqrt{3}+\frac{1}{\sqrt{2}}}{\frac{1}{2}\sqrt{3}-\frac{1}{\sqrt{2}}}}$

**(c)** $\dfrac{1}{\sqrt{12}-\sqrt{140}}-\dfrac{1}{\sqrt{8}-\sqrt{60}}-\dfrac{2}{\sqrt{10}+\sqrt{80}}$

**Ans.** **(a)** $\sqrt{3}+5$ **(b)** $\sqrt{3}+\sqrt{2}$ **(c)** $0$